MMP Learning Seminar

Week 96.

Contents:

- log discrepancies.
- exceptional singularities.

Log discrepancies:

(X,Δ) log pair, $\Delta \geq 0$, $K_X+\Delta$ is $\mathbb{Q}$-Cartier.

$$\begin{array}{l} Y \xrightarrow[\varphi]{} X \quad \text{proj bir} \\ \cup\!\text{I} \\ E \\ \text{prime divisor.} \end{array}$$

$$\varphi^*(K_X+\Delta) = K_Y+\Delta_Y.$$

The log discrepancy of (X,Δ) at E is $1-\mathrm{coeff}_E(\Delta_Y)$.

Minimal log discrepancy: $(X,\Delta;x)$ → "a point on X".

$$\mathrm{mld}(X,\Delta;x) = \inf\left\{ a_E(X,\Delta) \;\middle|\; c_X(E)=x \right\}.$$

Example: $\mathrm{mld}(\mathbb{A}^n;\{0\}) = n.$

$\mathrm{mld}(C_n;\{0\}) = \frac{2}{n}.$

Cone over a rational curve of degree n

Remark: Minimal log discrepancies are harder to compute than log canonical thresholds.

minimizing a function on a "lattice".

minimizing a function on a convex set.

Conjecture (ACC): The set of n-dimensional minimal log discrepancies satisfies the ACC.

Conjecture (LSC): The minimal log discrepancy function is lower semicontinuous on the closed point x.

Known cases:

LSC:
- up to dimension 3.
- LCI (deFernex - Ein - Mustăţă)
- Quotient sing (Nakamura).

ACC:
- ✓ Surface sing (Alexeev - Shokurov).
- ✓ Toric sing (Borisov - Ambro).
- ✓ Quotient sing (Nakamura - M).
- ✓ 3-fold near 1 (Prokhorov - Chen).
- ✓ exceptional (M, Shokurov - Han - Li).
- regularity one. (M)

classification or structure

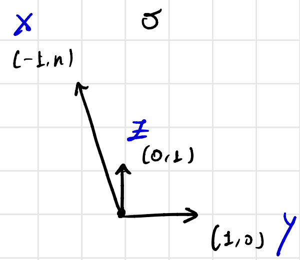
use BAB & theory of complements.

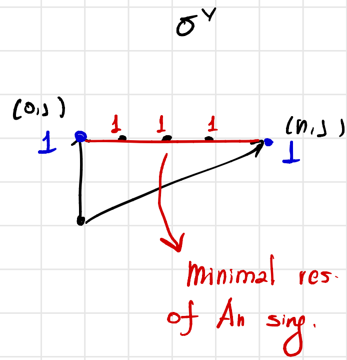
mld > 1

Terminal 3-fold singularities are classified by Reid

Toric singularity:

$X_n = \{ xy + z^n = 0 \}$



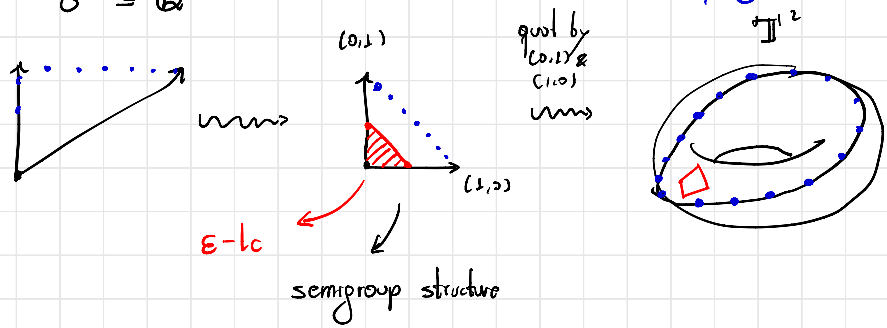


$x + y = nz$

Theorem: The log discrepancy function of $X(\sigma)$ correspond to the unique linear function L_σ on $\sigma^\vee$ that value. 1. on $\underbrace{\sigma^\vee(1)}_{\text{lattice ray gen}}$

$\sigma^\vee \subseteq \mathbb{Q}^2$

$X(\sigma) \rightsquigarrow$



ε-lc

Theorem (Lawrence, 90's): Let $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$.

$U \subseteq \mathbb{T}^d$ open set. The set of subgroups of $\mathbb{T}^d$ which do not intersect U has only finitely many maximal elements wrt inclusion

Remark: A terminal 3-fold singularity (X, x) of index r admits $r-1$ blow-ups extracting divisors with log discrepancy $1 + \frac{\alpha}{r}$ where $\alpha \in \{1, \ldots, r-1\}$

Q: What happens in the case of terminal toric 3-fold sing?

$X_r \longleftarrow$ terminal toric 3-fold of index r

$\rightsquigarrow$

associated cone

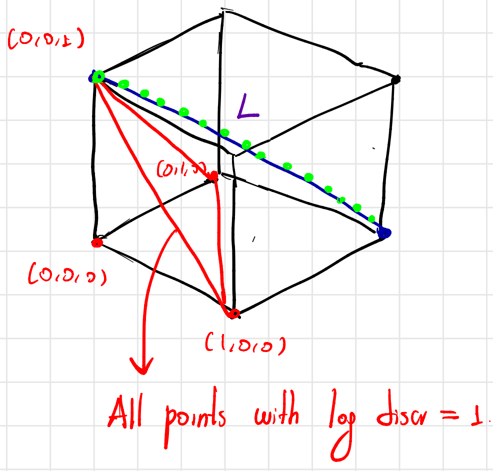
$\rightsquigarrow$

associated semigroup in $\mathbb{R}^3_{\geq 0}$

$\rightsquigarrow$

associated group in $\mathbb{T}^3$.

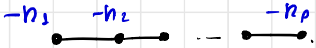
$L \subseteq \mathbb{T}^3$.



X_r corresponds to

$L \cap \mathbb{Z}\left[\frac{1}{r}\right] \subseteq \mathbb{T}^3$

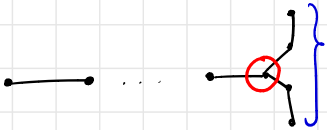
Surface klt singularities: classified by Alexeev.

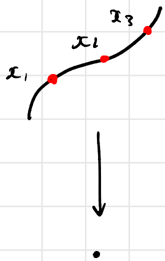
A-type: 

$n_i \geq 2$. $\Longleftrightarrow$ toric surface singularity.

D-type: 

$n_i \geq 2$. $\Longleftrightarrow$ admit 2:1 cover making them toric.

E-type:  at most 5 nodes.



x_1, x_2 & x_3 are toric sing.

$(X; x)$ E-type surface

Quotient singularities:

Theorem (Jordan, 1890's): Let $(X;x)$ be a n-dimensional quotient singularity. There exists a n-dimensional toric singularity $(T;t)$ and a finite Galois morphism
$$\begin{aligned} T &\longrightarrow X \\ t &\longmapsto x. \end{aligned}$$
of degree at most $c(n)$.

Remark: If n is large enough, then we can take $c(n) = n!$
$n \geq 71$.

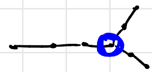
Regularity of klt singularities:

$(X,\Gamma;x)$ is a log canonical sing. We define its regularity to be

$$\mathrm{reg}\,(X,\Gamma;x) = \dim \{ \mathcal{D}(Y,\Gamma_Y) \mid (Y,\Gamma_Y) \longrightarrow (X,\Gamma) \text{ dlt mod} \}$$

$(X,\Delta;x)$ klt pair. We define its regularity to be.

$$\mathrm{reg}\,(X,\Delta;x) = \max \{ \mathrm{reg}(X,\Gamma;x) \mid (X,\Gamma;x) \text{ is lc } \& \ \Gamma \geq \Delta \}.$$

$\left.\begin{array}{l} A\text{-type} \\ D\text{-type} \end{array}\right\}$ $\mathrm{reg} = 1.$ E-type $\}$ $\mathrm{reg} = 0$ 

dihedral.

Example: Toric singularities have regularity $= \dim X - 1$.

Example: There are quotient sing of $\mathrm{reg} = 0$.

Definition: A singularity is said to be exceptional if

$$\mathrm{reg} = 0.$$

For instance, E_6, E_7 & E_8 are exceptional sing.

Exceptional singularities:

Proposition: A two dimensional quotient singularity $X = \mathbb{C}^2/G$ by a finite group G without reflections is exceptional if and only if G has no semi-invariants of degree ≤ 2.

(same result holds in dimension 3 with semi-invariants of degree ≤ 3).

Remark: 4-dimensional & 5-dimensional quotient exceptional sing have been classified by Prokhorov and Shramov.

Remark: Let $X \subseteq \mathbb{C}^4$ be a hypersurface canonical singularity given by the equation:

$$x_1^{a_1} + \dots + x_4^{a_4} = 0.$$

Prokhorov & Ishii classified all the weights $(a_1, \dots, a_4)$ for which this is an exceptional sing.

Theorem (Han-Liu-M, 2019): If we fix $n \in \mathbb{Z}_{\geq 1}$ & $\varepsilon > 0$

The set of n-dim ε-lc exceptional singularities are bounded up to deformation.

Theorem (M, 2018): The ACC for mlds holds for exceptional sing.

Proof: $(X;x)$ n-dim, exceptional & ε-lc. $(\varepsilon > 0)$.

By boundedness of complements (Birkar), we can find a lc N-complement $(X, \Gamma; x)$ is lc

$$N(K_X + \Gamma) \sim 0 \quad \text{around } x$$

Let $Y \supseteq E$ be the unique log canonical place.

X' is the index one cover of K_X.

$$\begin{tikzcd}
Y' \supseteq E' \arrow[r] \arrow[d] & E \subseteq Y \arrow[d] \\
X' \ni x' \arrow[r] & x \in X
\end{tikzcd}$$

ε-lc, n-dim, exc.

obtained by adj to E

$N(K_E + \Gamma_E) \sim 0$ $\quad$ (E, Γ_E) is klt.

$N(K_{E'} + \Gamma_{E'}) \sim 0$ $\quad$ $(E', \Gamma_{E'})$ is klt

$\xrightarrow{\text{BAB}}$ belong to a bounded family

E & E' are $\frac{1}{N}$-lc and Fano.

In particular, the degree of $E' \to E$ is bounded above.

$$\begin{tikzcd} Y' \supseteq E' \arrow[r] \arrow[d, "\pi'"'] & E \subseteq Y \arrow[d] \\ X' \ni x' \arrow[r] & x \in X \end{tikzcd}$$

We can find $C \subseteq E'$ lying on the smooth loci for which $C \cdot \Gamma_{E'} \leq k$

$$K_{Y'} + \Gamma_{Y'} + E' \sim_{\mathbb{Q}, X'} 0.$$ pull-back of $K_{X'} + \Gamma'$

$$K_{Y'} + (1 - a_{E'}(X'))E' \sim_{\mathbb{Q}, X'} 0$$ pull back of $K_{X'}$

$$a_{E'}(X')E' + \Gamma_{Y'} \sim_{\mathbb{Q}, X'} 0.$$ / product C.

$$a_{E'}(X')(-m) + (\underbrace{\Gamma_{Y'} \cdot C}_{\leq k}) = 0.$$

$a_{E'}(X') \leq \frac{k}{m}$ is bounded above.

$$\frac{k}{m} \geq a_{E'}(X') = (\text{ramification at } E)\, \underset{\textstyle \vee \atop \textstyle \varepsilon}{a_E(X)}$$

Riemann - Hurwitz.

$a_E(X) \leq k/m$ $\quad \boxed{\mathrm{mld}(X; x) \leq k}$

$$\frac{k}{m} \geq \alpha_{E'}(X') = (\text{ramification at } E)\ \underset{\underset{\varepsilon}{\vee}}{\alpha_E(X)}$$

ramification index $\lesssim \dfrac{k}{\varepsilon m}$ ↖ bounded above in terms of the dim

$$\text{Degree}\,(Y' \longrightarrow Y) = \text{Degree}\,(E' \longrightarrow E) \times \underbrace{\text{ramification index at } E}_{\text{bounded above.}}$$

$\|$

index of K_X.

We conclude that the index of K_X is bounded above,

$i(K_X) \mid m$.

$\alpha_E(X) \in \mathbb{Z}\left[\frac{1}{\ell}\right]$. $\qquad \text{mld}(X; x) \leq \dfrac{k}{m}$.

$$\text{mld}(X; x) \in \underbrace{\left[0, k/m\right] \cap \mathbb{Z}\left[\tfrac{1}{\ell}\right]}_{\text{finite}}$$

$\blacksquare$